1. $f(x) = \sqrt{1+x}$; $f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}}$; $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$; $f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-\frac{5}{2}}$; $f'''(0) = \frac{3}{8}$.

$$p_3(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0)$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

Now $\sqrt{0.75} = \sqrt{1-0.25} = f(-0.25)$ $\left(\because f(x) = \sqrt{1+x}\right)$

$\therefore$ We approximate $\sqrt{0.75}$ by $p_3(-0.25)$

$$p_3(-0.25) = 1 + \frac{1}{2}(-0.25) - \frac{1}{8}(-0.25)^2 + \frac{1}{16}(-0.25)^3$$
$$\simeq 0.866210938$$

$\therefore \left| p_3(-0.25) - \sqrt{0.75} \right| \simeq 0.000185534$

<u>Ans</u> (a) If $p_3(x) = A + Bx + Cx^2 + Dx^3$, then

$A = 1$

$B = \frac{1}{2}$

$C = -\frac{1}{8}$

$D = \frac{1}{16}$

(b) $p_3(-0.25) = 0.866210938$

(c) $|\text{error}| = \left| p_3(-0.25) - \sqrt{0.75} \right| \simeq 0.000185534$.

Marks

a) $1+1+1+1 = 4$

b) 1

c) 1

Total : $\boxed{6 \text{ marks}}$

2. $f(x) = e^x$

$f^{(n)}(x) = e^x \ \forall n$; $f^{(n)}(0) = 1 \ \forall n$.

$$\therefore P_4(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Now $R_4(x) = e^x - P_4(x) = \dfrac{f^{(5)}(\xi)x^5}{5!} = \dfrac{e^{\xi} x^5}{120}$, $\quad 0 < \xi(x) < x$. and ξ depends on x.

$$\therefore |R_4(-0.5)| = \frac{e^{\xi(-0.5)} \left|(-0.5)^5\right|}{120} \leq \frac{(0.5)^5}{120}(1) = 0.0002604166$$

(since $\xi(-0.5)$ lies between 0 and -0.5 and note that $\xi(-0.5) \leq 0$ and thus $e^{\xi(-0.5)} \leq e^0 = 1$.)

The absolute error is

$$\left|P_4(-0.5) - e^{-0.5}\right| = \left|1 + (-0.5) + \frac{(-0.5)^2}{2} + \frac{(-0.5)^3}{6} + \frac{(-0.5)^4}{24} - e^{-0.5}\right|$$

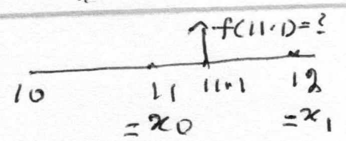
$$\simeq 0.0002402103$$

Marks
(a) 2
(b) 2
$\boxed{\text{Total: 4 marks}}$

Ans: (a) $|R_4(-0.5)| \leq 0.0002604166$

(b) $\left|P_4(-0.5) - e^{-0.5}\right| \simeq 0.0002402103$

3. We want to approximate $f(x)$ by a linear interpolating polynomial that interpolates $f(x)$ at $(x_0, f(x_0)) = (11, 65)$ and $(x_1, f(x_1)) = (12, 69)$.



$f(x_0) = f(11) = 65$; $f(x_1) = f(12) = 69$.

$$\therefore f(11.1) \simeq P_1(11.1) = f(11) + \frac{x - 11}{12 - 11}(69 - 65) \quad \left(\because f(x) = f(x_0) + p\Delta f_0;\ p = \frac{x - x_0}{h}\right)$$

$$= 65 + \frac{0.1}{1}(4)$$

$$= 65 + 0.4 = 65.4.$$

Marks : 2
$\boxed{\text{Total marks: 2}}$

Ans:
$\therefore f(11.1) \simeq P_1(11.1) = \underline{65.4}$.

4. Forward difference table.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
.40	$-.916291$ $=f(x_0)$				
		.223144 $=\Delta f_0$			
.50	$-.693147$		$-.040823$ $=\Delta^2 f_0$		
		.182321		.012653 $=\Delta^3 f_0$	
.60	$-.510826$		$-.028170$		$-.005103$ $=\Delta^4 f_0$
		.154151		.00755	
.70	$-.356675$		$-.020620$		
		.133531			
.80	$-.223144$				

$$p = \frac{x - x_0}{h} = \frac{.54 - .40}{.10} = \frac{.14}{.10} = 1.4.$$

$$f(x) \simeq P_4(x) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0)$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f(x_0),$$ Taking $x = 0.54$,

$$P_4(.54) = -.916291 + (1.4)(.223144) + \frac{(1.4)(.4)}{2}(-.040823)$$
$$+ \frac{(1.4)(0.4)(-0.6)}{6}(0.012653) + \frac{(1.4)(.4)(-.6)(-1.6)}{24}(-.005103)$$
$$= -.916291 + .3124016 - 0.01143044 - .00070857 - .00011431$$
$$= -.61614273.$$

$\therefore \ln(.54) \simeq P_4(.54) = -.61614273.$

Ans:

a) The fourth order forward difference $\Delta^4 f(x_0) = \underline{-.005103}$.

b) $\ln(.54) \simeq P_4(.54) = \underline{-.61614273}$.

Marks

a) 5 marks

b) 3 marks

$\boxed{\text{Total: 8 marks}}$

5. Using 10 decimal digit accuracy

$x - y = 0.0001248121$.

Now, both x and y are rounded to five decimal digits before subtraction

$x \simeq 0.37215 = x^*$

$y \simeq 0.37202 = y^*$

$\therefore x^* - y^* = 0.00013$

$\therefore$ relative error $= \dfrac{|(x-y) - (x^*-y^*)|}{|x-y|} \simeq 0.04158321$

Ans: a) $x - y = .0001248121$

b) If x and y are rounded to five decimal digits x^* and y^* respectively, then

$x^* = 0.37215$

$y^* = 0.37202$

$x^* - y^* = 0.00013$.

c) relative error $= .04158321$

$\simeq 4.16\%$.

Marks:

(a) 1

(b) 1+1+1 = 3

(c) 1

$\boxed{\text{Total: 5 marks}}$